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ANALYTICAL SOLUTION FOR CORRELATED STOCHASTIC PROCESSES

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Abstract. *We develop an analytical solution for the Laplace Transform of a non necessarily uncorrelated two-factor diffusion process integral. We exhibit the exponential affine solution of the Laplace transform calculated via Riccati differential equations. In the simulations carried out, the Laplace transform solution tracks the solution of a two correlated CIR models inferred via Monte Carlo. The result encounters applications in two real-world situations: pricing bonds via splitting the nominal interest rates as a combination of real interest rates and actual inflation, and calculating the failure probability of a engineering system.*

Keywords: *Affine Processes, Correlation, Laplace Transform*

1. INTRODUCTION

The stochastic models found in finance and engineering applications usually assume uncorrelation among state variables. It follows that the square-root diffusion process ((1)) miss the analytical treatment when the stochastic factors are subject to correlation between the Brownian motions pairs. This stochastic differential equations ensure mean reversion of the state variables towards a long run level and avoids the possibility of negative values of the process. These are interesting properties for a number of practical applications, especially when the two processes are correlated. The correlation creates difficult even when numerical solutions of the corresponding partial differential equations are given by finite difference or finite element methods.

In this paper we present a two-factor mean-reversion correlated process which reduces to the square-root diffusion when the correlation is set to zero. We show that the Laplace transform of the vector integrated process has analytical solution of exponential affine form. We solve the Laplace transform by using the Feynman-Kac formula, transforming the expectation into a partial differential equation problem. The solution of the PDE is breached by using the separation of variables approach, resulting in a set of Riccati Equations, which have known explicit solutions.

The numerical results show negligible discrepancies between the formula and the Monte Carlo simulation of the correlated square-root diffusion. When the processes are uncorrelated, the discrepancy is equal to zero. This fact can be veryfied both theoretically and numerically.

We introduce the model and its Laplace transform closed-form solution in section 2. In section 3 we show that the Laplace transform of the two-factor integrated stochastic process finds real-world applications in the context of finance and reliability engineering. In section 4 we present some numerical issues. We present an interpretation of the Laplace transform when highly positive and negative correlations are considered in the failure probability problem.

2. The Model

Let Y be a vector of 2 state variables given by

$$dx = \kappa_x(\theta_x - x)dt + \sigma_x\sqrt{x}dZ_x, \tag{1}$$

$$dr = \kappa_r(\theta_r - r)dt + \sigma_r\left(\frac{\rho\epsilon}{\sqrt{x}}dZ_x + \sqrt{(1-\rho^2)r - \left(\frac{\rho\epsilon}{\sqrt{x}}\right)^2}dZ_r\right), \tag{2}$$

where κ is the speed of adjustment, θ is the mean, σ is the volatility and ϵ is a constant. The parameter ρ is the correlation between the standard Wiener processes Z_x and Z_r.

Equation (1) is a Feller process or the square-root diffusion process known in the context of finance as the CIR process due to the authors Cox, Ingersoll and Ross ((1)). The following conditions must hold for the processes (1) and (2) in order to avoid the possibility of negative and complex values of x_t and r_t, respectively:

$$2\kappa_x\theta_x \geq \sigma_x^2 \quad \text{and} \quad \frac{(1-\rho^2)r_t x_t}{\rho^2} \geq \epsilon^2.$$

Theorem 2..1 *Let Y be the vector process given by the set of stochastic differential equations* (1) *and* (2). *The closed form solution of the Laplace Transform of the integrated processes* (1) *and* (2) *is given by*

$$f(Y_t, t, T) = \mathbb{E}[e^{-\int_t^T \delta' Y_u du}|Y_t] = \exp(A(t,T) - B(t,T)x(t) - C(t,T)r(t)), \tag{3}$$

where

$$A(t,T) = \int_t^T -\kappa_x\theta_x B(t) - \kappa_r\theta_r C(t) + \rho\epsilon\sigma_x\sigma_r B(t)C(t)dt, \tag{4}$$

$$B(t,T) = \frac{2(e^{\gamma_x(T-t)} - 1)}{(\gamma_x + \kappa_x)(e^{\gamma_x(T-t)} - 1) + 2\gamma_x}, \tag{5}$$

$$C(t,T) = \frac{2(e^{\gamma_r(T-t)} - 1)}{(\gamma_r + \kappa_r)(e^{\gamma_r(T-t)} - 1) + 2\gamma_r}, \tag{6}$$

$$\gamma_x = \sqrt{\kappa_x^2 + 2\sigma_x}, \tag{7}$$

$$\gamma_r = \sqrt{\kappa_r^2 + 2\sigma_r(1-\rho^2)}, \tag{8}$$

$$\delta = \mathbf{1} \in \mathbb{R}^{2\times 1}. \tag{9}$$

Proof. Following (2), the Laplace Trasform of a multifactor stochastic differential equation of the form

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dZ_t,$$

has exponential affine solution if the coefficients of Y are of the form

$$\mu(y) = K_0 + K_1 y \quad \text{and} \quad (\sigma(y)\sigma(y)')_{ij} = H_{0ij} + H_{1ij}y.$$

We see that if Y is given by (1) and (2), then $K_0 = [\kappa_x\theta_x; \kappa_r\theta_r]'$, $K_1 = \begin{bmatrix} -\kappa_x & 0 \\ 0 & -\kappa_r \end{bmatrix}$ and

$$(\sigma(y)\sigma(y)') = \begin{bmatrix} \sigma_x^2 x & \rho\sigma_x\sigma_r\epsilon \\ \rho\sigma_x\sigma_r\epsilon & \sigma_r^2(1-\rho^2)r \end{bmatrix}$$

is of the affine form.

Applying the Feyman-Kac formula to the Discounted Laplace Transform function results in the following PDE

$$\frac{\partial f(y,t)}{\partial t} + \frac{\partial f(y,t)}{\partial y}(K_0 + K_1 y) + \sum_{i,j} \frac{\partial f(y,t)}{\partial y_i \partial y_j}(H_{0ij} + H_{1ij}y) = (\delta')yf(y,t).$$

Combining the coefficients of Y and the exponential affine function of the solution to the Discounted Laplace Transform leads to the the following partial differential equation

$$\begin{aligned} \frac{\partial A}{\partial t} + x\frac{\partial B}{\partial t} + r\frac{\partial C}{\partial t} + [\kappa_x(\theta_x - x)]B + [\kappa_r(\theta_r - r)]C + \frac{1}{2}\sigma_x^2 x B^2 \\ + \frac{1}{2}\sigma_r^2 r C^2(1-\rho^2) + \rho\sigma_x\sigma_r BC\epsilon = \delta' y(t), \end{aligned}$$

Gathering the terms in x and in r gives

$$\frac{\partial A}{\partial t} = -\kappa_x\theta_x B - \kappa_r\theta_r C - \rho\sigma_x\sigma_r BC\epsilon \tag{10}$$

and the following Riccati Differential Equations:

$$\frac{\partial B}{\partial t} = -\kappa_x B - \frac{\sigma_x^2}{2}B^2 + 1 \tag{11}$$

$$\frac{\partial C}{\partial t} = -\kappa_r C - \frac{\sigma_r^2}{2}C^2(1-\rho^2) + 1. \tag{12}$$

Given the terminal conditions

$$A(T,T) = B(T,T) = C(T,T) = 0,$$

the solutions of the Riccati Equations of the form (11) and (12) are developed in (3). They are exhibited in (5), (6), (7) and (8). □

It is worth mentioning that the two-factor uncorrelated CIR process given by

$$dx = \kappa_x(\theta_x - x)dt + \sigma_x\sqrt{x}dZ_x, \tag{13}$$

$$dr = \kappa_r(\theta_r - r)dt + \sigma_r\sqrt{r}dZ_r, \tag{14}$$

introduced in the famous paper (1) which closed form solution is shown in (4), is a particular case of the analytical solution (3) when $\rho = 0$.

3. Applications

3.1 Reliability engineering of correlated devices in a serial system

The probability function of the arrival of random events is

$$\mathbb{P}[X = x] = \frac{\lambda^x (T-t)^x e^{-\lambda(T-t)}}{x!}, \tag{15}$$

known as Poisson distribution. Here λ is a deterministic mean rate of arrivals between time t and T. If τ is the time of a random event we have its distribution function given by

$$F(t,T) \;=\; \mathbb{P}[t \leq \tau \leq T] = 1 - e^{-\lambda(T-t)}. \tag{16}$$

It can be shown ((5) and (6)) that when the mean rate λ is a stochastic process, called intensity process, then

$$\mathbb{P}[\tau > T] = \mathbb{E}\left[exp\left(-\int_t^T \lambda(s)ds\right)\right]. \tag{17}$$

It follows that the solution of (17) is the same as that of a Discounted Laplace Transform. Setting $Y = \lambda$ we can model the failure event as an exponential distribution with stochastic mean rate λ.

Suppose Y is a vector of two arrival failure rates associated with a series reliability system. In this particular case, the failure rate of one component may vary if the other component degrades. Let the pair $[r_t, x_t]$ denote such components of failure rate and ρ the associated correlation.

Then, the solution (3) gives the reliability of the system and

$$\mathbb{P}[t \leq \tau \leq T] = 1 - f(Y_t, t, T)$$

gives the failure probability of the system. The Mean Time To Failure, the length of time the system is expected to last in operation, is given by

$$MTTF = \int_0^\infty f(Y_t, t, T)dt.$$

3.2 Bond pricing with decomposed Nominal Yields

Following (7), nominal yields y_t of zero-coupon bonds can be decomposed into real yield r_t and inflation premium x_t. Then,

$$y_t = r_t + x_t \tag{18}$$

is an approximation of the famous Fisher equation (1930).

We can see that if we allow real yield and inflation premium to be two correlated processes, the price of a zero-coupon bond is given by solving the Discounted Laplace Transform function (3). The solution (3) is therefore the present value of the zero-coupon bond which certainly pays 1 at maturity T.

4. Numerical Results

We found that high positive or high negative correlated process results in lower values for the Laplace Transform than the non-correlated case. The difference across high or low and zero correlations become more prominent as times to maturities increase. For example, for $T = 20$ years and the following parameters

$$\kappa = (0.5; 0.5) \quad \theta = (0.1; 0.1) \quad \sigma = (0.15; 0.15)$$

we found a relative discrepancy of about 6.5% between them. Figure 1 illustrates the relation between the Laplace transform values and the correlation. This non-monotony feature is highlighted in the context of finance in (9).

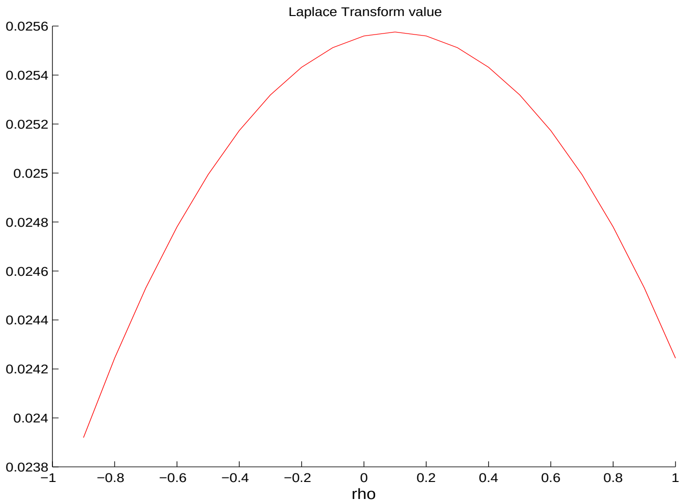

Figure 1- Laplace Transform.

When

$$\epsilon = h(1 - \rho^2)x_0 r_0$$

, the Laplace transform solutions given by (3) results in a maximum relative discrepancy of 0.04% of the numerical solution of the correlated CIR model calculated via Monte Carlo simulations when $T = 3$ for $0 \leq h \leq 1$. Setting $T = 10$, the relative discrepancy increases to no more than 3.8%.

Using the interpretation presented in subsection 3.1, the failure probability of a serial system is minimum at zero correlation between component 1 and 2. A decrease in correlation to the negative field means a higher probability of either component 1 or 2 failing. In the case of a improvement in component 1, the failure rate of component 2 increases causing a lower reliability of the system. If the correlation increases the system is due to fail if either component 1 or 2 failure rates increase, diminishing the reliability of the system.

In figure 2 we produced an example to show that the solutions of (3) converge when $h \to 0$ for a fixed correlation.

5. Conclusions

The paper developed an analytical solution for the Laplace Transform of two correlated stochastic processes via Riccati differential Equations. The result is intended to calculate im-

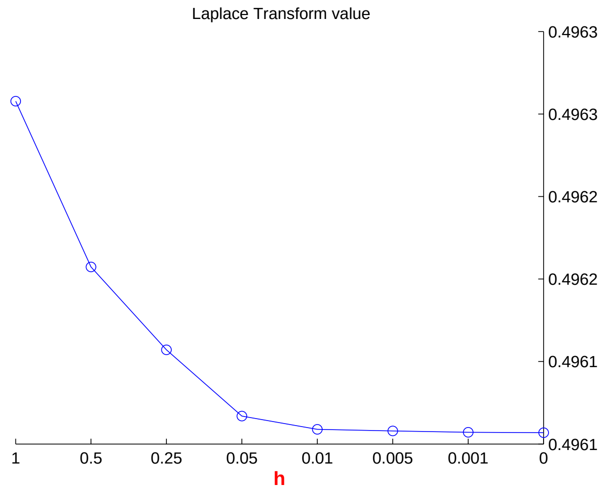

Figure 2- Test with ϵ.

portant metrics of reliability engineering and to price bonds splitting the nominal interest rates as a combination of real interest rates and actual inflation.

The results are a very good approximation to the correlated case when compared to Monte Carlo simulation results of the CIR model. Moreover, if the correlation parameter is set equal to zero the results match when compared to the non-correlated formula.

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