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A TOPOLOGICAL DERIVATIVE-BASED METHOD FOR AN INVERSE POTENTIAL PROBLEM MODELED BY A MODIFIED HELMHOLTZ EQUATION

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Abstract. *This paper deals with an inverse problem whose forward model is governed by a modified Helmholtz equation. The inverse problem consists in the reconstruction of a set of anomalies embedded into a geometrical domain from partial measurements of a scalar field of interest taken on the boundary of the reference domain. Since the inverse problem, we are dealing with, is written in the form of an ill-posed boundary value problem, the idea is to rewrite it as a topology optimization problem. In this scenario, we use the concept of topological derivatives. Hence, the shape functional measuring the misfit between the known target data and the calculated data is minimized with respect to a set of ball-shaped inclusions. It leads to a non-iterative reconstruction algorithm which is independent of any initial guess. As a result, the reconstruction process becomes very robust with respect to the noisy data. A numerical example is presented in order to demonstrate the effectiveness of the proposed algorithm.*

Keywords: *Inverse potential problem, modified Helmholtz equation, higher order topological derivatives, topology optimization, reconstruction method.*

1 INTRODUCTION

In this paper, we study an inverse potential problem in $\mathbb{R}^2$ whose corresponding forward problem is governed by a modified Helmholtz equation. The inverse problem under consideration is about the reconstruction of a set of anomalies embedded into a geometrical domain with the help of measurements of the associated potential. For this purpose, let us consider $\Omega \subset \mathbb{R}^2$ an open and bounded domain with smooth boundary $\partial\Omega$ where the measurements of the associated potential are collected. We represent each anomaly by an open and connected set $\omega_i^* \subset \Omega$ such that $\overline{\omega_i^*} \cap \overline{\omega_j^*} = \emptyset$ for $i \neq j$ and $\overline{\omega_i^*} \cap \partial\Omega = \emptyset$ for each $i, j \in \{1, \ldots, N^*\}$ with $N^* \in \mathbb{Z}^+$ the number of anomalies. If $\omega^* = \cup_{i=1}^{N^*} \omega_i^*$, then, for $i = 1, \ldots, N^*$, ω_i^* is an open connected component of ω^*. See sketch in Fig. 1(a).

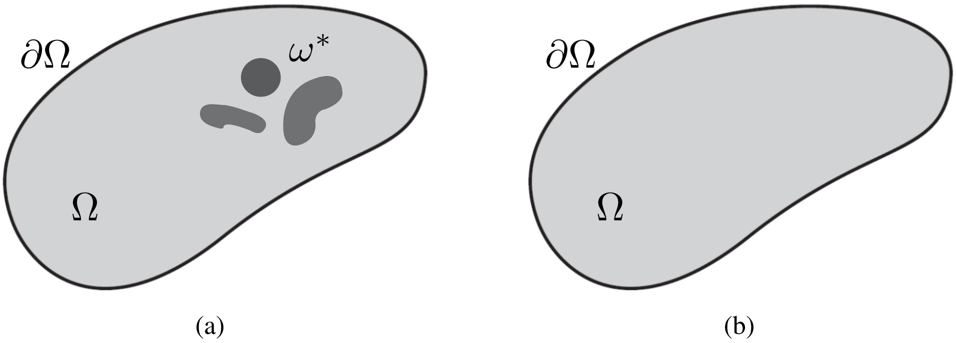


Figure 1- Domain Ω (a) with and (b) without a set of anomalies ω^*.

We consider the domain Ω as a bounded region representing a fluid medium which contains a different fluid substance within a subdomain ω^*. The inverse problem consists in finding k_{ω^*} such that the substance concentration z satisfies the following over-determined boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta z + k_{\omega^*} z & = & 0 & \text{in} \quad \Omega, \\ \partial_n z & = & g_N & \text{on} \quad \partial\Omega, \\ z & = & g_D & \text{on} \quad \partial\Omega, \end{array} \right. \tag{1}$$

where g_N and g_D are the boundary excitation and boundary measurement, respectively. The parameter k_{ω^*} is defined as

$$k_{\omega^*} = \left\{ \begin{array}{ll} k & \text{in } \Omega \setminus \overline{\omega^*}, \\ \gamma_i k & \text{in } \omega_i^*, \ i = 1, \ldots, N^*, \end{array} \right. \tag{2}$$

with $k, \gamma_i \in \mathbb{R}^+$, where γ_i is the contrast with respect to the material property of the background k. Now, for an initial guess k_ω of k_{ω^*}, we consider the substance concentration field u to be the solution to the boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u + k_\omega u & = & 0 & \text{in} \quad \Omega, \\ \partial_n u & = & g_N & \text{on} \quad \partial\Omega, \end{array} \right. \tag{3}$$

where

$$k_\omega = \left\{ \begin{array}{ll} k & \text{in } \Omega \setminus \overline{\omega}, \\ \gamma_i k & \text{in } \omega_i, \ i = 1, \ldots, N. \end{array} \right. \tag{4}$$

Since we want our guess k_ω to be close to the unknown k_{ω^*}, it is natural to wish that the scalar fields u and z have the same measurement on $\partial\Omega$. Keeping this objective in mind, we rewrite our inverse problem in the form of the following topology optimization problem given by

$$\underset{\omega \subset \Omega}{\text{Minimize}} \ \mathcal{J}_\omega \left(u^1, \cdots, u^M \right) = \sum_{m=1}^{M} \int_{\partial\Omega} \left(u^m - z^m \right)^2 , \tag{5}$$

where $M \in \mathbb{Z}^+$ is the number of observations, z^m and u^m are the solutions of the boundary value problems (1) and (3), respectively, corresponding to the Neumann data g_N^m with $m = 1, \cdots, M$. Notice that, the minimizer of the topology optimization problem (5) produces the best approximation to ω^*, solution of the inverse problem (1), in an appropriate sense. In addition, it is well known that we can not reconstruct both the support ω_i^* and the associated contrast γ_i simultaneously (Fernandez et al., 2018). Therefore, we assume that the contrast γ_i, for $i = 1, \ldots, N^*$, is known and we focus on the reconstruction of their supports ω_i^* from the boundary measurements taken on $\partial\Omega$.

Such type of inverse problem under investigation in this work can be solved by using iterative and non-iterative methods. Among a variety of methods, we want to draw the attention of the readers on the level-set method and the methods based on asymptotic expansions such as those devised from the topological derivative concept (Sokołowski & Żochowski, 1999). More recently, some reconstruction problems have been solved with the help of higher order topological derivatives which allow the development of non-iterative methods that are independent on the initial guess (Canelas et al., 2014; Ferreira & Novotny, 2017; Machado et al., 2017; Rocha & Novotny, 2017). Iterative algorithms based on level-set methods are widely used to solve inverse reconstruction problems (Burger et al., 2004; Doel et al., 2010; Isakov et al., 2011). In contrast to the methods based on the topological derivatives, level-set-based methods are dependent on the initial guess and, in general, the reconstruction process requires a high number of iterations. Following the original ideas presented in Fernandez et al. (2018), we use higher order topological derivatives to solve the inverse problem of interest of this paper.

The remaining part of this paper is organized as follows. The proposed inverse problem is to be solved by using the concept of topological derivatives, therefore, we present in Section 2 the shape functionals corresponding to the unperturbed and perturbed domains as well as the asymptotic expansion of some Bessel functions. In Section 3, we obtain the higher-order topological expansion of the shape functional. The non-iterative reconstruction algorithm is devised in Section 4 and a numerical example of the reconstruction of multiple anomalies is presented in Section 5. Conclusions are discussed in Section 6.

2 PRELIMINAIRES

In this section, we introduce some tools related to the theory of topological derivatives as well as the asymptotic expansions of some Bessel functions. For an account on the topological derivative concept, the reader may refer to the book by Novotny & Sokołowski (2013).

2.1 Topology optimization setting

The inverse problem (1) has been written in the form of a topology optimization problem (5). The basic idea is minimizing the shape functional appearing in (5) by using the topological derivatives. For this purpose, we introduce the shape functionals related to the unperturbed and perturbed domains. Since the topological derivative does not depend on the initial guess of the unknown topology ω^*, we start with the unperturbed domain by setting $\omega = \emptyset$ (see Fig. 1(b)). More precisely, we consider

$$\mathcal{J}_0\left(u_0^1, \cdots, u_0^M\right) = \sum_{m=1}^{M} \int_{\partial\Omega} \left(u_0^m - z^m\right)^2, \tag{6}$$

where u_0^m be the solution of the unperturbed boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u_0^m + k u_0^m & = & 0 & \text{in} \quad \Omega, \\ \partial_n u_0^m & = & g_N^m & \text{on} \quad \partial\Omega, \end{array} \right. \tag{7}$$

In this paper, we are considering the topology optimization problem (5) for the ball-shaped anomalies and hence we define the topologically perturbed counter-part of (6) by introducing $N \in \mathbb{Z}^+$ number of small circular inclusions $B_{\varepsilon_i}(x_i)$ with center at $x_i \in \Omega$ and radius ε_i for $i = 1, \cdots, N$. The set of inclusions can be denoted as $B_\varepsilon(\xi) = \cup_{i=1}^N B_{\varepsilon_i}(x_i)$, where $\xi = (x_1, \ldots, x_N)$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$. Moreover, we assume that $\overline{B_\varepsilon} \cap \partial\Omega = \emptyset$ and $\overline{B_{\varepsilon_i}(x_i)} \cap \overline{B_{\varepsilon_j}(x_j)} = \emptyset$ for each $i \neq j$ and $i, j \in \{1, \cdots, N\}$. The shape functional associated with the topologically perturbed domain is written as

$$\mathcal{J}_\varepsilon\left(u_\varepsilon^1, \cdots, u_\varepsilon^M\right) = \sum_{m=1}^{M} \int_{\partial\Omega} \left(u_\varepsilon^m - z^m\right)^2 \tag{8}$$

with u_ε^m be the solution of the perturbed boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u_\varepsilon^m + k_\varepsilon u_\varepsilon^m & = & 0 & \text{in} \quad \Omega, \\ \partial_n u_\varepsilon^m & = & g_N^m & \text{on} \quad \partial\Omega, \end{array} \right. \tag{9}$$

where the parameter k_ε is defined as

$$k_\varepsilon = \left\{ \begin{array}{ll} k & \text{in } \Omega \setminus \cup_{i=1}^N \overline{B_{\varepsilon_i}(x_i)}, \\ \gamma_i k & \text{in } B_{\varepsilon_i}(x_i), \ i = 1, \ldots, N. \end{array} \right. \tag{10}$$

2.2 Series expansions for Bessel functions

In this section, we introduce the asymptotic expansion of some modified Bessel functions to be used next. We denote the modified Bessel functions of the first kind and order n by I_n with $n \in \mathbb{Z}$. As $x \to 0^+$, we have the following asymptotic expansions:

$$I_0(x) = 1 + \frac{1}{4}x^2 + \tilde{I}_0(x) \quad \text{and} \quad I_1(x) = \frac{1}{2}x + \frac{1}{16}x^3 + \tilde{I}_1(x) \tag{11}$$

with $\tilde{I}_0(x) = O(x^4)$ and $\tilde{I}_1(x) = O(x^5)$. The modified Bessel functions of the second kind and order n are denoted by K_n with $n \in \mathbb{Z}$. As $x \to 0^+$, we have the following asymptotic expansions:

$$K_0(x) = (\ln 2 - \zeta) - \ln x - \frac{1}{4}x^2 \ln x + \frac{1}{4}(1 + \ln 2 - \zeta)x^2 + \tilde{K}_0(x), \quad \tilde{K}_0(x) = O(x^4) \tag{12}$$

and

$$K_1(x) = \frac{1}{x} + \frac{1}{2}x \ln x + \frac{1}{2}\left(\zeta - \ln 2 - \frac{1}{2}\right)x + \frac{1}{16}x^3 \ln x + \frac{1}{16}\left(\zeta - \ln 2 - \frac{5}{4}\right)x^3 + \tilde{K}_1(x), \tag{13}$$

$\tilde{K}_1(x) = O(x^5)$. In (12) and (13), ζ is the Euler constant. The above series expansions were obtained from the book by Jeffrey (2004).

3 TOPOLOGICAL ASYMPTOTIC EXPANSIONS

Here, we first propose an *ansätz* for the asymptotic expansion of u_ε^m, solution of the problem related to the topologically perturbed domain. Next, by using this expansion for u_ε^m, it is possible to proceed with the asymptotic expansion of the shape functional $\mathcal{J}_\varepsilon\left(u_\varepsilon^1, \cdots, u_\varepsilon^M\right)$ from which the reconstruction scheme is devised.

3.1 Asymptotic expansion of the solution

Firstly, let us introduce the quantity $\beta_i = \gamma_i - 1$ and the vector $\alpha \in \mathbb{R}^N$, where each entry α_i denotes the Lebesgue measure (volume) of the two-dimensional ball $B_{\varepsilon_i}(x_i)$, i.e,

$$\alpha = (\alpha_1, \cdots, \alpha_N) \quad \text{with} \quad \alpha_i = |B_{\varepsilon_i}(x_i)| = \pi\varepsilon_i^2, \qquad \text{for} \quad i = 1, \cdots, N. \tag{14}$$

The perturbed shape functional $\mathcal{J}_\varepsilon\left(u_\varepsilon^1, \cdots, u_\varepsilon^M\right)$, given by (8), depends on the small parameter ε through the solution u_ε^m of the problem (9), for $m = 1, \ldots, M$. Therefore, we propose the following *ansätz* for the expansion of u_ε^m with respect to the parameters corresponding to the small circular inclusions as described in Section 2.1

$$u_\varepsilon^m(x) = u_0^m(x) + k\sum_{i=1}^N \alpha_i\beta_i h_i^{\varepsilon,m}(x) + k^2\sum_{i=1}^N\sum_{j=1}^N \alpha_i\alpha_j\beta_i\beta_j h_{ij}^{\varepsilon,m}(x) + \tilde{u}_\varepsilon^m(x), \tag{15}$$

where, for each $i, j = 1, \cdots, N$ and $m = 1, \cdots, M$, $h_i^{\varepsilon,m}$, $h_{ij}^{\varepsilon,m}$ and $\tilde{u}_\varepsilon^m$ are the solutions of

$$\left\{\begin{array}{rcll} -\Delta h_i^{\varepsilon,m} + k h_i^{\varepsilon,m} &=& -(\alpha_i)^{-1} u_0^m \chi_{B_{\varepsilon_i}(x_i)} & \text{in} \quad \Omega, \\ \partial_n h_i^{\varepsilon,m} &=& 0 & \text{on} \quad \partial\Omega, \end{array}\right. \tag{16}$$

$$\left\{\begin{array}{rcll} -\Delta h_{ij}^{\varepsilon,m} + k h_{ij}^{\varepsilon,m} &=& -(\alpha_i)^{-1} h_j^{\varepsilon,m} \chi_{B_{\varepsilon_i}(x_i)} & \text{in} \quad \Omega, \\ \partial_n h_{ij}^{\varepsilon,m} &=& 0 & \text{on} \quad \partial\Omega, \end{array}\right. \tag{17}$$

and

$$\left\{\begin{array}{rcll} -\Delta \tilde{u}_\varepsilon^m + k_\varepsilon \tilde{u}_\varepsilon^m &=& \Phi_\varepsilon^m & \text{in } \Omega, \\ \partial_n \tilde{u}_\varepsilon^m &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{18}$$

respectively. In problem (18), we have $\Phi_\varepsilon^m = -k^3\sum_{i,j,l=1}^N \alpha_j\alpha_l\beta_i\beta_j\beta_l h_{jl}^{\varepsilon,m}\chi_{B_{\varepsilon_i}(x_i)}$. In order to simplify further analysis, we write $h_i^{\varepsilon,m}$ as a sum of three functions p_i^ε, q_i and $\tilde{h}_i^{\varepsilon,m}$ in the form

$$h_i^{\varepsilon,m} = u_0^m(x_i)h_i^\varepsilon + \tilde{h}_i^{\varepsilon,m} \quad \text{with} \quad h_i^\varepsilon = p_i^\varepsilon + q_i^\varepsilon. \tag{19}$$

The function p_i^ε is solution of

$$\left\{\begin{array}{rcll} -\Delta p_i^\varepsilon + k p_i^\varepsilon &=& -(\alpha_i)^{-1}\chi_{B_{\varepsilon_i}(x_i)} & \text{in } B_R(x_i), \\ p_i^\varepsilon &=& \lambda_3(\varepsilon_i)K_0(\sqrt{k}R) & \text{on } \partial B_R(x_i), \end{array}\right. \tag{20}$$

with $B_{\varepsilon_i}(x_i) \subset \Omega \subset B_R(x_i)$, $x_i \in \Omega$, $0 < \varepsilon_i \ll R$. Moreover, $\lambda_3(\varepsilon_i)$ denotes a constant depending on ε_i to be presented next. Problem (20) can be solved analytically and its solution is

$$p_i^\varepsilon(x) = \left\{\begin{array}{ll} \lambda_1(\varepsilon_i) + \lambda_2(\varepsilon_i)I_0(\sqrt{k}\|x - x_i\|) & \text{in } B_{\varepsilon_i}(x_i), \\ \lambda_3(\varepsilon_i)K_0(\sqrt{k}\|x - x_i\|) & \text{in } B_R(x_i) \setminus \overline{B_{\varepsilon_i}(x_i)}, \end{array}\right. \tag{21}$$

with $\lambda_1(\varepsilon_i) = -\dfrac{1}{k\pi\varepsilon_i^2}$, $\lambda_2(\varepsilon_i) = \dfrac{1}{k\pi\varepsilon_i^2}\dfrac{K_1(\sqrt{k}\varepsilon_i)}{K_0(\sqrt{k}\varepsilon_i)I_1(\sqrt{k}\varepsilon_i) + K_1(\sqrt{k}\varepsilon_i)I_0(\sqrt{k}\varepsilon_i)}$ and $\lambda_3(\varepsilon_i) = -\dfrac{1}{k\pi\varepsilon_i^2}\dfrac{I_1(\sqrt{k}\varepsilon_i)}{K_0(\sqrt{k}\varepsilon_i)I_1(\sqrt{k}\varepsilon_i) + K_1(\sqrt{k}\varepsilon_i)I_0(\sqrt{k}\varepsilon_i)}$. Observe that we can use the asymptotic expansions (11)-(13) to rewrite $\lambda_2(\varepsilon_i)$ and $\lambda_3(\varepsilon_i)$ as follows

$$\lambda_2(\varepsilon_i) = \frac{1}{k\pi\varepsilon_i^2} + \lambda + \frac{1}{2\pi}\ln\varepsilon_i + \tilde{\lambda}_2(\varepsilon_i) \qquad \text{and} \qquad \lambda_3(\varepsilon_i) = -\frac{1}{2\pi} + \tilde{\lambda}_3(\varepsilon_i) \tag{22}$$

with $\lambda = \dfrac{1}{4\pi}(2\zeta + \ln k - 2\ln 2 - 1)$, $\tilde{\lambda}_2(\varepsilon_i) = O(\varepsilon_i^2)$ and $\tilde{\lambda}_3(\varepsilon_i) = O(\varepsilon_i^2)$. Taking into account the solution p_i^ε of the problem (20), we write $q_i^\varepsilon = \lambda_3(\varepsilon_i)q_i$, where q_i is the solution to the homogeneous boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta q_i + kq_i & = & 0 & \text{in } \Omega, \\ \partial_n q_i & = & -\partial_n K_0(\sqrt{k}\|x - x_i\|) & \text{on } \partial\Omega \end{array} \right. \tag{23}$$

and $\tilde{h}_i^{\varepsilon,m}$ solves the boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta \tilde{h}_i^{\varepsilon,m} + k\tilde{h}_i^{\varepsilon,m} & = & -(\alpha_i)^{-1}\left(u_0^m - u_0^m(x_i)\right)\chi_{B_{\varepsilon_i}(x_i)} & \text{in } \Omega, \\ \partial_n \tilde{h}_i^{\varepsilon,m} & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{24}$$

From the decomposition (19) and the solution of the problem (20), we can introduce the notation

$$h_i^{\varepsilon,m} := \left\{ \begin{array}{ll} u_0^m(x_i)h^\varepsilon_{i|_{B_{\varepsilon_i}}} + \tilde{h}_i^{\varepsilon,m} & \text{in } B_{\varepsilon_i}(x_i), \\ u_0^m(x_i)h^\varepsilon_{i|_{\Omega\setminus\overline{B_{\varepsilon_i}}}} + \tilde{h}_i^{\varepsilon,m} & \text{in } \Omega \setminus \overline{B_{\varepsilon_i}(x_i)}, \end{array} \right. \tag{25}$$

with $h^\varepsilon_{i|_{B_{\varepsilon_i}}} := p^\varepsilon_{i|_{B_{\varepsilon_i}}} + \lambda_3(\varepsilon_i)q_i$ and $h^\varepsilon_{i|_{\Omega\setminus\overline{B_{\varepsilon_i}}}} := p^\varepsilon_{i|_{\Omega\setminus\overline{B_{\varepsilon_i}}}} + \lambda_3(\varepsilon_i)q_i$, where $p^\varepsilon_{i|_{B_{\varepsilon_i}}}$ is the solution of the problem (20) in $B_{\varepsilon_i}(x_i)$ and $p^\varepsilon_{i|_{\Omega\setminus\overline{B_{\varepsilon_i}}}}$ is the solution of the same problem in $\Omega \setminus \overline{B_{\varepsilon_i}(x_i)}$. Moreover, we also introduce an adjoint state v^m as the solution of the following auxiliary boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta v^m + kv^m & = & 0 & \text{in } \Omega, \\ \partial_n v^m & = & u_0^m - z^m & \text{on } \partial\Omega. \end{array} \right. \tag{26}$$

3.2 Asymptotic expansion of the shape functional

From the *ansätz* for u_ε^m given by (15), we can obtain the asymptotic expansion of the shape functional $\mathcal{J}_\varepsilon\left(u_\varepsilon^1, \cdots, u_\varepsilon^M\right)$ with respect to α. Let us start considering the difference between the perturbed shape functional $\mathcal{J}_\varepsilon\left(u_\varepsilon^1, \cdots, u_\varepsilon^M\right)$ and its unperturbed counter-part $\mathcal{J}_0\left(u_0^1, \cdots, u_0^M\right)$ defined in (8) and (6), respectively, which yields to the following simplified expression

$$\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) = \sum_{m=1}^M \int_{\partial\Omega} \left[2\left(u_\varepsilon^m - u_0^m\right)\left(u_0^m - z^m\right) + \left(u_\varepsilon^m - u_0^m\right)^2\right]. \tag{27}$$

By using (15) in (27), we get

$$\begin{aligned} \mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) \;=\;& 2k\sum_{m=1}^M\sum_{i=1}^N \alpha_i\beta_i \int_{\partial\Omega} h_i^{\varepsilon,m}\left(u_0^m - z^m\right) \\ &+ 2k^2 \sum_{m=1}^M\sum_{i=1}^N\sum_{j=1}^N \alpha_i\alpha_j\beta_i\beta_j \int_{\partial\Omega} h_{ij}^{\varepsilon,m}\left(u_0^m - z^m\right) \\ &+ k^2 \sum_{m=1}^M\sum_{i=1}^N\sum_{j=1}^N \alpha_i\alpha_j\beta_i\beta_j \int_{\partial\Omega} h_i^{\varepsilon,m}h_j^{\varepsilon,m} + o(|\alpha|^2). \end{aligned} \tag{28}$$

Now, let us introduce the weak formulation of the adjoint problem (26) which is to find $v^m \in H^1(\Omega)$ such that

$$\int_\Omega \nabla v^m \cdot \nabla \eta + k\int_\Omega v^m\eta = \int_{\partial\Omega}\left(u_0^m - z^m\right)\eta, \quad \forall \eta \in H^1(\Omega). \tag{29}$$

The weak formulations of the problems (16) and (17) are to find $h_i^{\varepsilon,m} \in H^1(\Omega)$ such that

$$\int_\Omega \nabla h_i^{\varepsilon,m} \cdot \nabla \eta + k \int_\Omega h_i^{\varepsilon,m} \eta = -\frac{1}{\alpha_i} \int_{B_{\varepsilon_i}(x_i)} u_0^m \eta, \quad \forall \eta \in H^1(\Omega) \tag{30}$$

and $h_{ij}^{\varepsilon,m} \in H^1(\Omega)$ such that

$$\int_\Omega \nabla h_{ij}^{\varepsilon,m} \cdot \nabla \eta + k \int_\Omega h_{ij}^{\varepsilon,m} \eta = -\frac{1}{\alpha_i} \int_{B_{\varepsilon_i}(x_i)} h_j^{\varepsilon,m} \eta, \quad \forall \eta \in H^1(\Omega), \tag{31}$$

respectively. By taking $\eta = h_i^{\varepsilon,m}$ in (29) and $\eta = v^m$ in (30) as test functions, we get

$$\int_{\partial\Omega} h_i^{\varepsilon,m} (u_0^m - z^m) = -\frac{1}{\alpha_i} \int_{B_{\varepsilon_i}(x_i)} u_0^m v^m. \tag{32}$$

Similarly, if we take $\eta = h_{ij}^{\varepsilon,m}$ in (29) and $\eta = v^m$ in (31) as test functions, it gives

$$\int_{\partial\Omega} h_{ij}^{\varepsilon,m} (u_0^m - z^m) = -\frac{1}{\alpha_i} \int_{B_{\varepsilon_i}(x_i)} h_j^{\varepsilon,m} v^m. \tag{33}$$

Thus, by replacing (32) and (33) into (28), we get

$$\begin{aligned} \mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) &= -2k \sum_{m=1}^M \sum_{i=1}^N \beta_i \int_{B_{\varepsilon_i}(x_i)} u_0^m v^m - 2k^2 \sum_{m=1}^M \sum_{i=1}^N \sum_{j=1}^N \alpha_j \beta_i \beta_j \int_{B_{\varepsilon_i}(x_i)} h_j^{\varepsilon,m} v^m \\ &\quad + k^2 \sum_{m=1}^M \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \beta_i \beta_j \int_{\partial\Omega} h_i^{\varepsilon,m} h_j^{\varepsilon,m} + o(|\alpha|^2). \end{aligned} \tag{34}$$

Finally, we rewrite the expansion (34) in its final form by considering (a) the decomposition of the function $h_i^{\varepsilon,m}$ introduced in Section 3.1 together with the analytical form of p_i^ε and the asymptotic expansions for $\lambda_1(\varepsilon_i)$, $\lambda_2(\varepsilon_i)$ and $\lambda_3(\varepsilon_i)$ as well as (b) the Taylor's expansions of the functions u_0^m, v^m and $p^\varepsilon_{j|_{\Omega\setminus\overline{B_{\varepsilon_j}}}}$ around the point x_i. Proceeding in this way, we obtain

$$\begin{aligned} \mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}_0(u_0) &= -2k \sum_{m=1}^M \sum_{i=1}^N \alpha_i \beta_i u_0^m(x_i) v^m(x_i) - \frac{1}{2\pi} k^2 \sum_{m=1}^M \sum_{i=1}^N \alpha_i^2 \ln \alpha_i \, \beta_i^2 u_0^m(x_i) v^m(x_i) \\ &- \frac{1}{2\pi} k^2 \sum_{m=1}^M \sum_{i=1}^N \alpha_i^2 \beta_i u_0^m(x_i) v^m(x_i) - \frac{1}{2\pi} k \sum_{m=1}^M \sum_{i=1}^N \alpha_i^2 \beta_i \nabla u_0^m(x_i) \cdot \nabla v^m(x_i) \\ &- \frac{1}{4\pi} \sigma k^2 \sum_{m=1}^M \sum_{i=1}^N \alpha_i^2 \beta_i^2 u_0^m(x_i) v^m(x_i) + \frac{1}{\pi} k^2 \sum_{m=1}^M \sum_{i=1}^N \alpha_i^2 \beta_i^2 u_0^m(x_i) v^m(x_i) q_i(x_i) \\ &+ \frac{1}{\pi} k^2 \sum_{m=1}^M \sum_{i=1}^N \sum_{\substack{j=1 \\ j\neq i}}^N \alpha_i \alpha_j \beta_i \beta_j u_0^m(x_j) v^m(x_i) \mathcal{K}_{ij} + \frac{1}{\pi} k^2 \sum_{m=1}^M \sum_{i=1}^N \sum_{\substack{j=1 \\ j\neq i}}^N \alpha_i \alpha_j \beta_i \beta_j u_0^m(x_j) v^m(x_i) q_j(x_i) \\ &+ \frac{1}{4\pi^2} k^2 \sum_{m=1}^M \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \beta_i \beta_j u_0^m(x_i) u_0^m(x_j) \mathcal{I}_{ij} + o(|\alpha|^2), \end{aligned} \tag{35}$$

where we have the constant $\sigma = -1 + 4\zeta + \ln \dfrac{k^2}{16\pi^2}$, the number $\mathcal{K}_{ij} = K_0(\sqrt{k}\|x_i - x_j\|)$ and the integral $\mathcal{I}_{ij} = \int_{\partial\Omega} [K_0(\sqrt{k}\|x - x_i\|) + q_i(x)][K_0(\sqrt{k}\|x - x_j\|) + q_j(x)]$.

4 RECONSTRUCTION ALGORITHM

The resulting non-iterative reconstruction algorithm based on the expansion (35) is described in this section. The topological asymptotic expansion of the shape functional $\mathcal{J}(u_\varepsilon)$ given by (35) can be rewritten in the following matrix equation

$$\mathcal{J}\left(u_\varepsilon\right) = \mathcal{J}_0\left(u_0\right) - \alpha \cdot d(\xi) + G(\xi)\alpha \cdot \operatorname{diag}(\alpha \otimes \log \alpha) + \frac{1}{2} H(\xi)\alpha \cdot \alpha + o(|\alpha|^2)\ , \quad (36)$$

where the vector $d \in \mathbb{R}^N$ and the matrices $G, H \in \mathbb{R}^{N\times N}$ have the following entries

$$d_i := 2k\beta_i \sum_{m=1}^{M} u_0^m(x_i) v^m(x_i), \quad (37)$$

$$G_{ii} := -\frac{1}{2\pi} k^2 \beta_i^2 \sum_{m=1}^{M} u_0^m(x_i) v^m(x_i), \quad G_{ij} = 0, \quad \text{if} \quad i \neq j \quad (38)$$

and

$$\begin{aligned} H_{ii} := & -\frac{1}{\pi} k^2 \beta_i \sum_{m=1}^{M} u_0^m(x_i) v^m(x_i) - \frac{1}{\pi} k\beta_i \sum_{m=1}^{M} \nabla u_0^m(x_i) \cdot \nabla v^m(x_i) - \frac{1}{2\pi} \sigma k^2 \beta_i^2 \sum_{m=1}^{M} u_0^m(x_i) v^m(x_i) \\ & + \frac{2}{\pi} k^2 \beta_i^2 \sum_{m=1}^{M} u_0^m(x_i) v^m(x_i) q_i(x_i) + \frac{1}{2\pi^2} k^2 \beta_i^2 \sum_{m=1}^{M} [u_0^m(x_i)]^2\, \mathcal{I}_{ii}, \quad (39) \end{aligned}$$

$$\begin{aligned} H_{ij} \ := \ & \frac{2}{\pi} k^2 \beta_i \beta_j \sum_{m=1}^{M} u_0^m(x_j) v^m(x_i) \mathcal{K}_{ij} + \frac{2}{\pi} k^2 \beta_i \beta_j \sum_{m=1}^{M} u_0^m(x_j) v^m(x_i) q_j(x_i) \\ & + \frac{1}{2\pi^2} k^2 \beta_i \beta_j \sum_{m=1}^{M} u_0^m(x_i) u_0^m(x_j)\, \mathcal{I}_{ij}, \quad (40) \end{aligned}$$

if $i \neq j$; respectively, for $i, j = 1, \cdots, N$.

Note that the expression on the right-hand side of Eq. (35) depends explicitly on the number N of anomalies, their positions ξ and sizes α. Thus, let us now introduce the quantity

$$\delta J(\alpha, \xi, N) := -\alpha \cdot d(\xi) + G(\xi)\alpha \cdot \operatorname{diag}(\alpha \otimes \log \alpha) + \frac{1}{2} H(\xi)\alpha \cdot \alpha. \quad (41)$$

After minimizing (41) with respect to α, we obtain the following non-linear system

$$\left(H(\xi) + G(\xi)\right)\alpha + 2G(\xi)\operatorname{diag}(\alpha \otimes \log \alpha) = d(\xi) \quad (42)$$

which it is solved by using Newton's method. Observe that, if the quantity α is solution of the mentioned system, then it becomes a function of the locations ξ, that is, $\alpha = \alpha(\xi)$. Now, let us replace the solution $\alpha = \alpha(\xi)$ of the Eq. (42) in Eq. (41), to obtain

$$\delta J(\alpha\left(\xi\right), \xi, N) := -\frac{1}{2}\left(d(\xi) + G(\xi)\alpha(\xi)\right) \cdot \alpha(\xi). \quad (43)$$

Therefore, the pair of vectors $(\xi^\star, \alpha^\star)$ which minimizes (41) is given by

$$\xi^\star := \underset{\xi \in X}{\operatorname{argmin}} \ \delta J(\alpha(\xi), \xi, N) \quad \text{and} \quad \alpha^\star := \alpha(\xi^\star), \tag{44}$$

where X is the set of admissible locations of the anomalies. Thus, the minimizer of (41) is a set of ball-shaped disjoint inclusions denoted by $\omega^\star$ which is completely characterized by the pair $(\xi^\star, \alpha^\star)$. The optimal locations $\xi^\star$ can be trivially obtained from a combinatorial search over all the n-points of the set X and the optimal sizes are given by the second expression in (44). In short, for a given number of inclusions N, our method is able to find in one step their sizes $\alpha^\star$ and their locations $\xi^\star$. For more sophisticated approaches based on meta-heuristic and multi-grid methods, we refer to Machado et al. (2017), where the algorithm proposed in this section can be found in pseudo-code format.

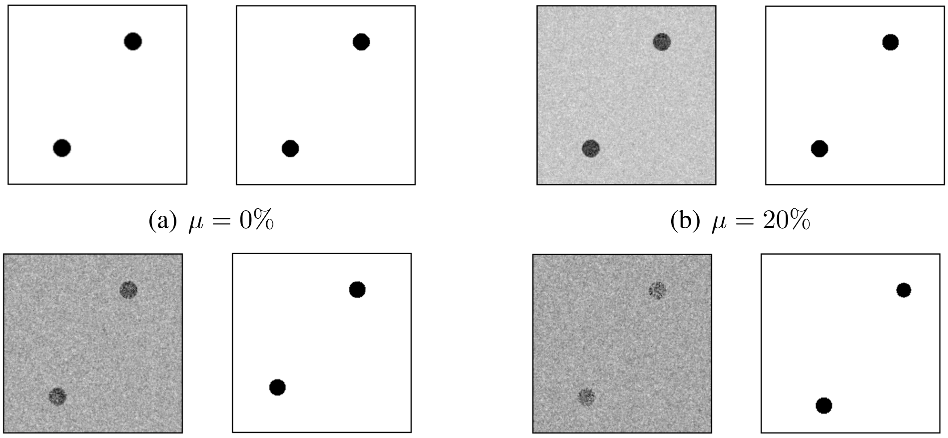
5 NUMERICAL RESULTS

We present a numerical example in order to test the reconstruction scheme proposed in the previous section. Given a target domain comprises a set of anomalies ω^*, our aim is to reconstruct the set ω^* by a set of circular disjoint inclusions from the help of measurements of a known scalar field z^m (related to the target domain), for $m = 1, \ldots, M$, taken on the boundary $\partial\Omega$. As previously commented, we desire to obtain the location ξ and the size α of each component of the target geometrical subdomain ω^* by assuming that the parameters k, $\gamma_i \in \mathbb{R}^+$, for $i = 1, \ldots, N$, are known. For simplicity, we set $k = 1$ and $\gamma_i = 2$, for $i = 1, \ldots, N$.

The geometric domain is a square $\Omega = (-0.5, 0.5) \times (-0.5, 0.5)$ which is discretized using three-node finite element scheme. The mesh is generated as a grid of 160×160 squares. Each square is divided into four triangles which leads to 102400 number of finite elements and 51521 nodes. However, due to the high complexity of the reconstruction algorithm, a sub-mesh is defined over the finite element mesh where the combinatorial search is performed, leading to the optimal solution $(\xi^\star, \alpha^\star)$ defined in the sub-mesh. Here, such sub-mesh comprises a total of 181 uniformly distributed nodes.

The proposed reconstruction scheme is also tested in the presence of noisy data. In this case, in order to obtain the noisy synthetic data, the parameter k_{ω^*} is replaced by $k^\mu_{\omega^*} = k_{\omega^*}(1 + \mu\tau)$, where τ is random variable taking values in the interval $(-1, 1)$ and μ corresponds to the noise level.

Reconstruction of two embedded anomalies is demonstrated in this example. As can be seen in Fig. 2(a)-(left), two circular regions with centers located at $x_1^* = (0.2, 0.3)$, $x_2^* = (-0.2, -0.3)$ and radii $\varepsilon_1^* = \varepsilon_2^* = 0.05$ are considered as the target anomalies. In the current setting, we take only one observation produced by the Dirichlet data $g = 1$ on the boundary $\partial\Omega$. The results obtained for different levels of noise are shown in Fig. 2. It can be seen in Fig. 2(a)-(right) that the anomalies are reconstructed accurately in the absence of noise. By comparing Figs. 2(b) and 2(c), we can observe that the reconstruction scheme works efficiently up to 50% of noise in the parameter k_{ω^*}. For more noisy input, though we are not sure about the accuracy but the functionality of the proposed scheme is ensured which can be observed in Fig. 2(d).



(a) $\mu = 0\%$ (b) $\mu = 20\%$

(c) $\mu = 50\%$ (d) $\mu = 70\%$

Figure 2- Target (left) and the respective result (right) for different levels of noise.

6 CONCLUSIONS

In this paper, a reconstruction method for solving an inverse problem modeled by a modified Helmholtz equation has been proposed. The reconstruction algorithm is devised from higher-order topological derivatives. In addition, the mentioned algorithm is non-iterative and independent of any initial guess. Hence, the reconstruction process becomes very robust with respect to the noisy. On the other hand, the approximation of the solution by a finite number of ball-shaped inclusions can be seen as a limitation of our approach. Despite this last fact, our approach can be used to produce a more accurate initial guess for iterative approaches such as the ones based on level-set methods.

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