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# STABILIZED HYBRID MIXED DGFEM FOR THE STOKES-DARCY PROBLEM APPLIED TO MISCIBLE DISPLACEMENTS IN HETEROGENEOUS MEDIA

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***Abstract.*** *In this work we apply the stabilized hybrid mixed finite element method developed and analyzed by Igreja and Loula (2018) to solve the incompressible miscible displacements in heterogeneous media formed by the coupling of the free-fluid with the porous medium. The hydrodynamic problem is governed by the Stokes and Darcy systems coupled by Beavers-Joseph-Saffman interface conditions. To solve the Stokes-Darcy coupled system we use the stabilized hybrid mixed method, characterized by the introduction of the Lagrange multipliers associated with the velocity field in both domains. The global system is assembled involving only the degrees of freedom associated with the multipliers and the variables of interest can be solved at the element level. Considering the velocity fields given by the hybrid method we adopted the SUPG method (Brooks and Hughes, 1982) combined with an implicit finite difference scheme to solve the transport equation associated with miscible displacements. Numerical studies are presented to illustrate the flexibility and robustness of the hybrid formulation. To verify the efficiency of the hybrid method, computer simulations are also presented for the recovery hydrological flow problems in heterogeneous porous media, such as continuous injection.*

***Keywords:*** *Finite Element Method, Hybrid Mixed Methods, Stokes-Darcy Flow, Coupled Problems, Heterogeneous Media*

## 1. INTRODUCTION

Numerical methods to simulate the incompressible viscous fluid flows coupling Stokes-Darcy problems has been widely developed due to various applications in physiological phenomena like the blood motion in vessels, hydrological systems in which surface water percolates through rocks and sand, petroleum engineering where are find fractured media containing vugs and caves as the naturally fractured carbonate karst reservoirs and industrial processes involving filtration (Hanspal et al., 2006; Arbogast and Brunson, 2007; Vassilev and Yotov, 2009; Núñez et al., 2015). This coupled problem is characterized by the coexistence of the free fluid governed by the Stokes equations and the porous medium modeled by the Darcy problem connected by the interface conditions that guarantee continuity of mass and momentum across the interface (Beavers and Joseph, 1967; Saffman, 1971).

Numerically, among the several methods proposed for the coupled problem, we highlight the stable and stabilized methods introduced in Salinger et al. (1994); Discacciati et al. (2002); Burman and Hansbo (2007); Masud (2007); Correa and Loula (2009), using a Lagrange multiplier to impose the interface restrictions we can cite Layton et al. (2003); Urquiza et al. (2008); Gatica et al. (2011) and employing discontinuous Galerkin (DG) methods Rivière (2005); Vassilev and Yotov (2009). Recently, hybridizations of DG methods have been successfully exploited to derive new finite element methods with improved stability and reduced computational cost but still preserving the robustness and flexibility of DG methods (Egger and Waluga, 2013; Igreja and Loula, 2018).

In this paper, in order to obtain efficiently the velocity field, we use the stabilized hybrid mixed method, to solve the coupled Stokes-Darcy problem developed and analyzed by Igreja and Loula (2018). This method is characterized by the introduction of the Lagrange multipliers associated with the velocity field to weakly impose continuity on each edge of the elements. Moreover, this method naturally imposes the interface conditions between porous medium and free fluid through the Lagrange multiplier. This methodology allow the elimination of the local problems at each element level in favor of the Lagrange multiplier. Thus, the system involves only global degrees of freedom associated with the multiplier, reducing significantly the computational cost. The accuracy of this method is presented through convergence studies.

Since calculated the hydrodynamic problem we supply the velocity field to the convection-dominated parabolic equation to obtain the concentrantion field in the coupled domain Stokes-Darcy. This results can, for example, characterizate a resevoir through the tracer injection processes, informing the preferred direction of flow (Malta et al., 2000; Núñez et al., 2015) or study the spread of pollution released in the water and assess the danger (Vassilev and Yotov, 2009). To illustrate the performance of the coupled Stokes-Darcy-transport problem, where the Streamline Upwind Petrov-Galerkin (SUPG) method (Brooks and Hughes, 1982) combined with a backward finite difference scheme in time is employed to approximate the concentration equation, is demonstrated via numerical simulations for the miscible transport problem using a five-spot pattern for different heterogeneous scenarios through continuous injection processes.

This paper is organized as follow. The model problem Stokes-Darcy-transport is introduced in Section 2. In Section 3, notations and definitions required to present the hybrid method. The stabilized mixed hybrid methods for the coupled Stokes-Darcy problem is presented in Section 4. The Section 5 is devoted to convergence study and continuous injection simulations. And finally, in Section 6, we present the concluding remarks of this work.

## 2. MODEL PROBLEM

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) the domain composed by two subdomains $\Omega_S$ and $\Omega_D$ related to free fluid and porous medium, respectively. In the subdomain $\Omega_S$, with outward unit normal $\mathbf{n}_S$, the flow is governed by the Stokes problem and in porous medium $\Omega_D$, with outward unit normal $\mathbf{n}_D$, the Darcy's law holds. These subdomains are separated by a smooth interface $\Gamma_{SD} = \partial\Omega_S \cap \partial\Omega_D$, where $\mathbf{t}_j$ define an orthonormal basis of tangent vectors on $\Gamma_{SD}$. Moreover, let $\Gamma = \Gamma_S \cup \Gamma_D$ with $\Gamma_i = \partial\Omega_i \setminus \Gamma_{SD}$ ($i = S, D$). The Figure 1 represents a sketch of the described domain.

Denoting $\mathbf{u}_i = \mathbf{u}|_{\Omega_i}$ and $p_i = p|_{\Omega_i}$, with $i = S, D$, the free fluid domain $\Omega_S$ is modeled by the Stokes problem that can be write as follow

*Given the viscosity $\mu$ and the source $\mathbf{f}$, find the pressure $p_S : \Omega_S \to \mathbb{R}$ and the velocity field*

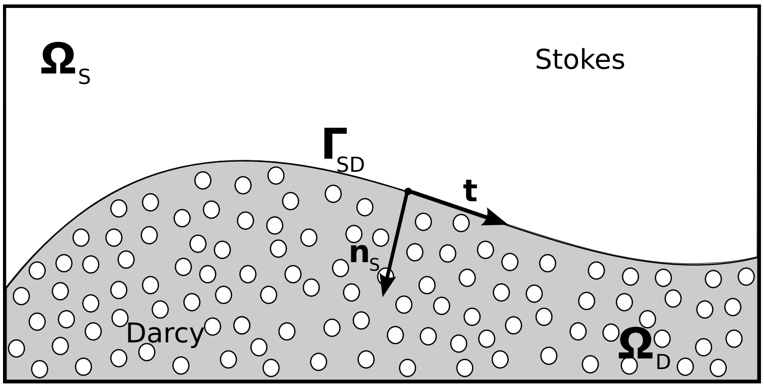



Figure 1- A sketch of coupled Stokes-Darcy domain.

$\mathbf{u}_S : \Omega_S \to \mathbb{R}^d$, *such that*

$$-\mu \operatorname{div} \nabla \mathbf{u}_S + \nabla p_S = \mathbf{f} \quad \text{in} \quad \Omega_S, \tag{1}$$
$$\operatorname{div} \mathbf{u}_S = 0 \quad \text{in} \quad \Omega_S, \tag{2}$$
$$\mathbf{u}_S = \mathbf{0} \quad \text{on} \quad \Gamma_S, \tag{3}$$

where div and $\nabla$ denote, respectively, the divergent and gradient operators. On the other hand, in porous medium the flow is given by the Darcy problem

*Given the hydraulic conductivity* $\mathbf{K}$ *and the source* $f$*, find the hydrostatic pressure* $p_D : \Omega_D \to \mathbb{R}$ *and the Darcy velocity* $\mathbf{u}_D : \Omega_D \to \mathbb{R}^d$*, such that*

$$\mathbf{u}_D = -\mathbf{K}\nabla p_D \quad \text{in} \quad \Omega_D, \tag{4}$$
$$\operatorname{div} \mathbf{u}_D = f \quad \text{in} \quad \Omega_D, \tag{5}$$
$$\mathbf{u}_D \cdot \mathbf{n}_D = 0 \quad \text{on} \quad \Gamma_D, \tag{6}$$

we define $\mathbf{K} = \mathbf{k}/\mu$ where $\mathbf{k}$ is the permeability of the porous medium and the solvability condition, which the source function $f$ must satisfy

$$\int_{\Omega_D} f \, d\mathbf{x} = 0.$$

On the interface free fluid/porous medium $\Gamma_{SD}$ are defined the following conditions

$$\mathbf{u}_S \cdot \mathbf{n}_S + \mathbf{u}_D \cdot \mathbf{n}_D = 0 \quad \text{on} \quad \Gamma_{SD}, \tag{7}$$
$$p_S - \mu \nabla \mathbf{u}_S \, \mathbf{n}_S \cdot \mathbf{n}_S = p_D \quad \text{on} \quad \Gamma_{SD}, \tag{8}$$
$$\mathbf{u}_S \cdot \mathbf{t}_j = -2\frac{\sqrt{\mathbf{k}}}{\alpha} \nabla \mathbf{u}_S \, \mathbf{n}_S \cdot \mathbf{t}_j, \quad j = 1, d-1, \quad \text{on} \quad \Gamma_{SD}. \tag{9}$$

The conditions (7) and (8) impose the continuity of flux and normal stress, respectively. The slip condition (9) is known as Beavers-Joseph-Saffman law (Beavers and Joseph, 1967; Saffman, 1971), where $\alpha > 0$ is an experimentally determined dimensionless constant. The coupled problem, modeled by the equations (1)-(9), is analyzed in detail by Layton et al. (2003), where existence and uniqueness of the solution is demonstrated.

The Stokes-Darcy coupled problem provides the velocity field for the diffusive-convective-reactive transport equation defined on the domain $\Omega = \Omega_S \cup \Omega_D$ whose problem is given by

*Given the velocity field* $\mathbf{u}$*, the porosity* $\phi$*, the sources* $\hat{f}$ *and* $g$ *and the function* $c_0$*, find the concentration* $c(\mathbf{x},t) : \Omega \times (0,T) \to \mathbb{R}^2$*, such that*

$$\phi \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c - \operatorname{div}(\mathbf{D}\nabla c) + \hat{f} c = g \qquad \text{in} \quad \Omega \times (0,T), \tag{10}$$

$$c(\mathbf{x},0) = c_0(\mathbf{x}) \qquad \text{in} \quad \Omega, \tag{11}$$

$$\mathbf{D}\nabla c \cdot \mathbf{n} = 0 \qquad \text{on} \quad \Gamma \times (0,T). \tag{12}$$

In the Stokes domain

$$\phi = 1 \quad \text{and} \quad \mathbf{D} = \alpha_m \mathbf{I} \quad \text{in} \quad \Omega_S, \tag{13}$$

where $\alpha_m$ is a molecular diffusion coeficient and $\mathbf{I}$ the identity tensor. In the porous medium, the tensor $\mathbf{D}$ can be defined as

$$\mathbf{D} = \mathbf{D}(\mathbf{u}_D) = \alpha_m \mathbf{I} + \|\mathbf{u}_D\| \left[\alpha_l \mathbf{E}(\mathbf{u}_D) + \alpha_t(\mathbf{I} - \mathbf{E}(\mathbf{u}_D))\right], \quad \mathbf{E}(\mathbf{u}) = \frac{\mathbf{u} \otimes \mathbf{u}}{\|\mathbf{u}\|^2},$$

with $\|\mathbf{u}\|^2 = u_1^2 + u_2^2$, $\otimes$ the tensorial product, $\alpha_l$ the longitudinal dispersion and $\alpha_t$ the transverse dispersion. In miscible displacement of a fluid through another in a reservoir the dispersion is physically more important than the molecular diffusion. Thus, we assume the following properties (Peaceman, 1977)

$$0 < \alpha_m \leq \alpha_l, \quad \alpha_l \geq \alpha_t > 0 \quad \text{and} \quad 0 < \phi \leq 1 \quad \text{in} \quad \Omega_D.$$

## 3. NOTATIONS AND DEFINITIONS

To introduce the stabilized hybrid formulation we first recall some notations and definitions. Let $H^m(\Omega)$ the usual Sobolev space equipped with the usual norm $\|\cdot\|_{m,\Omega} = \|\cdot\|_m$ and seminorm $|\cdot|_{m,\Omega} = |\cdot|_m$, with $m \geq 0$. For $m = 0$, we induction $L^2(\Omega) = H^0(\Omega)$ as the space of square integrable functions and $H_0^1(\Omega)$ the subspace of functions in $H^1(\Omega)$ with zero trace on $\partial\Omega$.

Restricting to the two-dimensional case ($d = 2$), we define a regular finite element partition $\mathcal{T}_h$ of the domain $\Omega$, as $\mathcal{T}_h = \{K\} :=$ the union of all elements $K$. In cases where $\Omega$ is divided into subdomains $\Omega_i$ with smooth boundary $\partial\Omega_i$ and $\Gamma_i = \partial\Omega \cap \partial\Omega_i$, we have for each subdomain the following regular partition $\mathcal{T}_h^i = \{K \in \mathcal{T}_h \cap \Omega_i\}$, and the following set of edges $\mathcal{E}_h^i = \{e; e \text{ is an edge of } K, \text{for at least one } K \in \mathcal{T}_h^i\}$, $\mathcal{E}_h^{\partial,i} = \{e \in \mathcal{E}_h^i; e \subset \Gamma_i\}$ $\mathcal{E}_h^{0,i} = \{e \in \mathcal{E}_h^i; e \text{ is an interior edge of } \Omega_i\}$, $\mathcal{E}_h^{ij} = \mathcal{E}_h^{0,i} \cap \mathcal{E}_h^{0,j}$. This last case denotes the edges that compose the interface between the subdomains, where $\Omega_i$ and $\Omega_j$ are two adjacent subdomains.

We assume that the domain $\Omega$ is polygonal. Thus, there exists $c > 0$ such that $h \leq ch_e$, where $h_e$ is the diameter of the edge $e \in \partial K$ and $h$, the mesh parameter, is the maximum element diameter. For each element $K$ we associate a unit normal vector $\mathbf{n}_K$. Let $\mathcal{V}_h^k$ and $\mathcal{Q}_h^l$ denote broken function spaces on $\mathcal{T}_h$ given by

$$\mathcal{V}_h^k = \{\mathbf{v} \in \mathbf{L}^2(\Omega); \mathbf{v}_h|_K \in [\mathbb{Q}_k(K)]^2, \ \forall K \in \mathcal{T}_h\}, \tag{14}$$

$$\mathcal{Q}_h^l = \{q \in L^2(\Omega); q_h|_K \in \mathbb{Q}_l(K), \ \forall K \in \mathcal{T}_h\}, \tag{15}$$

where $\mathbb{Q}_k(K)$ and $\mathbb{Q}_l(K)$ denote the space of polynomial functions of degree at most $k$ and $l$, respectively, on each variable. To introduce the hybrid methods we define the following spaces associated with the Lagrange multipliers

$$\mathcal{M}_h^m = \{\boldsymbol{\mu} \in \mathbf{L}^2(\mathcal{E}_h) : \boldsymbol{\mu}|_e = [p_m(e)]^2,\ \forall e \in \mathcal{E}_h^0,\ \boldsymbol{\mu}|_e = \mathbf{0},\ \forall e \in \mathcal{E}_h^\partial\}, \tag{16}$$

$$\mathcal{W}_h^m = \{\boldsymbol{\mu} \in \mathbf{L}^2(\mathcal{E}_h) : \boldsymbol{\mu}|_e = [p_m(e)]^2,\ \forall e \in \mathcal{E}_h^0,\ \boldsymbol{\mu}|_e \cdot \mathbf{n}_e = 0,\ \forall e \in \mathcal{E}_h^\partial\}, \tag{17}$$

Similarly, $p_m(e)$ is the space of polynomial functions of degree at most $m$ on an edge $e$.

## 4. HYBRID MIXED METHOD FOR THE STOKES-DARCY PROBLEM

Unlike the numerical methods employing Lagrange multipliers only in the interface free fluid/porous medium to solve the coupled problem (Layton et al., 2003; Gatica et al., 2011), Igreja and Loula (2018) developed a method with Lagrange multipliers in all domain and therefore the conditions on the interface are naturally imposed yielding a symmetric, robust and stable formulation. This formulation can be viewed below

*Find $[\mathbf{u}_h^i, p_h^i] \in \mathcal{V}_h^k(\Omega_i) \times \mathcal{Q}_h^l(\Omega_i)$, with $i = S, D$, and the Lagrange multipliers $\boldsymbol{\lambda}_h^S \in \mathcal{M}_h^m(\mathcal{E}_h^S)$ and $\boldsymbol{\lambda}_h^D \in \mathcal{W}_h^m(\mathcal{E}_h^D)$ such that, for all $[\mathbf{v}_h, q_h] \in \mathcal{V}_h^k(\Omega) \times \mathcal{Q}_h^l(\Omega)$ and $[\boldsymbol{\mu}_h, \boldsymbol{\mu}_h] \in \mathcal{M}_h^m(\mathcal{E}_h^S) \times \mathcal{W}_h^m(\mathcal{E}_h^D)$*

$$A_{SD}([\mathbf{u}_h^S, \mathbf{u}_h^D, p_h^S, p_h^D, \boldsymbol{\lambda}_h, \boldsymbol{\lambda}_h]; [\mathbf{v}_h, q_h, \boldsymbol{\mu}_h, \boldsymbol{\mu}_h]) = F_{SD}([\mathbf{v}_h, q_h]), \tag{18}$$

with

$$\begin{aligned}
&A_{SD}([\mathbf{u}_h^S, \mathbf{u}_h^D, p_h^S, p_h^D, \boldsymbol{\lambda}_h^S, \boldsymbol{\lambda}_h^D]; [\mathbf{v}_h, q_h, \boldsymbol{\mu}_h, \boldsymbol{\mu}_h]) = \\
&\quad A_S([\mathbf{u}_h^S, p_h^S, \boldsymbol{\lambda}_h^S]; [\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]) + A_D([\mathbf{u}_h^D, p_h^D, \boldsymbol{\lambda}_h^D]; [\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]) \\
&\quad + \sum_{K \in \mathcal{T}_h^S} \left[ \int_{\Gamma_{SD}} (p_h^S - \mu \nabla \mathbf{u}_h^S \, \mathbf{n}_S \cdot \mathbf{n}_S)(\mathbf{v}_h - \boldsymbol{\mu}_h) \cdot \mathbf{n}_S \, ds \right. \\
&\quad + \int_{\Gamma_{SD}} (q_h - \mu \nabla \mathbf{v}_h \, \mathbf{n}_S \cdot \mathbf{n}_S)(\mathbf{u}_h^S - \boldsymbol{\lambda}_h^D) \cdot \mathbf{n}_S \, ds + \int_{\Gamma_{SD}} \frac{\mu\alpha}{\sqrt{\mathbf{k}}} (\mathbf{u}_h^S \cdot \mathbf{t})(\mathbf{v}_h \cdot \mathbf{t}) \, ds \\
&\quad \left. + \beta_D \int_{\Gamma_{SD}} (\mathbf{u}_h^S - \boldsymbol{\lambda}_h^D) \cdot \mathbf{n}_S (\mathbf{v}_h - \boldsymbol{\mu}_h) \cdot \mathbf{n}_S \, ds \right],
\end{aligned} \tag{19}$$

$$F_{SD}([\mathbf{v}_h, q_h]) = F_S(\mathbf{v}_h) + F_D([\mathbf{v}_h, q_h]). \tag{20}$$

The bilinear and linear forms $A_S(\cdot,\cdot)$ and $F_S(\cdot)$ and $A_D(\cdot,\cdot)$ and $F_D(\cdot)$ for Stokes and Darcy problem, respectively, are defined as follow

$$\begin{aligned}
&A_S([\mathbf{u}_h^S, p_h^S, \boldsymbol{\lambda}_h]; [\mathbf{v}_h, q_h, \boldsymbol{\mu}_h^S]) - F_S(\mathbf{v}_h) = \sum_{K \in \mathcal{T}_h^S} \left[ \int_K \mu \nabla \mathbf{u}_h^S : \nabla \mathbf{v}_h \, d\mathbf{x} \right. \\
&- \int_K \operatorname{div} \mathbf{u}_h^S \, q_h \, d\mathbf{x} - \int_K p_h^S \, \operatorname{div} \mathbf{v}_h \, d\mathbf{x} - \int_{\partial K \setminus \Gamma_{SD}} \mu \nabla \mathbf{u}_h^S \, \mathbf{n}_K \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h) ds \\
&- \int_{\partial K \setminus \Gamma_{SD}} \mu \nabla \mathbf{v}_h \, \mathbf{n}_K \cdot (\mathbf{u}_h^S - \boldsymbol{\lambda}_h^S) ds + \int_{\partial K \setminus \Gamma_{SD}} p_h^S \, (\mathbf{v}_h - \boldsymbol{\mu}_h) \cdot \mathbf{n}_K \, ds \\
&+ \int_{\partial K \setminus \Gamma_{SD}} q_h \, (\mathbf{u}_h^S - \boldsymbol{\lambda}_h^S) \cdot \mathbf{n}_K \, ds + \beta_S \int_{\partial K \setminus \Gamma_{SD}} (\mathbf{u}_h^S - \boldsymbol{\lambda}_h^S) \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h) ds \\
&\left. - \int_K \mathbf{f} \cdot \mathbf{v}_h d\mathbf{x} \right] = 0,
\end{aligned} \tag{21}$$

and

$$
\begin{aligned}
A_D([\mathbf{u}_h^D, p_h^D, \boldsymbol{\lambda}_h^D]; [\mathbf{v}_h, q_h, \boldsymbol{\mu}_h]) - F_D([\mathbf{v}_h, q_h]) = \sum_{K \in \mathcal{T}_h^D} \Bigg[ \int_K \mathbf{A}\mathbf{u}_h^D \cdot \mathbf{v}_h \, d\mathbf{x} \\
- \int_K p_h^D \operatorname{div} \mathbf{v}_h \, d\mathbf{x} - \int_K \operatorname{div} \mathbf{u}_h^D \, q_h \, d\mathbf{x} + \int_{\partial K} p_h^D \, (\mathbf{v}_h - \boldsymbol{\mu}_h) \cdot \mathbf{n}_K \, ds \\
+ \int_{\partial K} q_h \, (\mathbf{u}_h^D - \boldsymbol{\lambda}_h^D) \cdot \mathbf{n}_K \, ds + \beta_D \int_{\partial K} A \, (\mathbf{u}_h^D - \boldsymbol{\lambda}_h^D) \cdot (\mathbf{v}_h - \boldsymbol{\mu}_h) \, ds \\
+ \delta_1 \int_K \mathbf{K}(\mathbf{A}\mathbf{u}_h^D + \nabla p_h^D) \cdot (\mathbf{A}\mathbf{v}_h + \nabla q_h) \, d\mathbf{x} \\
+ \delta_2 \int_K A \operatorname{div} \mathbf{u}_h^D \operatorname{div} \mathbf{v}_h \, d\mathbf{x} + \delta_3 \int_K \kappa \, \operatorname{rot}(\mathbf{A}\mathbf{u}_h^D) \operatorname{rot}(\mathbf{A}\mathbf{v}_h) \, d\mathbf{x} \\
- \delta_2 \int_K A \, f \, \operatorname{div} \mathbf{v}_h \, d\mathbf{x} + \int_K f \, q_h \, d\mathbf{x} \Bigg] = 0.
\end{aligned}
\tag{22}
$$

The Lagrange multipliers $\boldsymbol{\lambda}_S$ and $\boldsymbol{\lambda}_D$ are identified as the trace of the velocity field, $\mathbf{A} = \mathbf{K}^{-1}$, $\kappa = \|\mathbf{K}\|_\infty$, $A = \kappa^{-1}$, where $\|\cdot\|_\infty$ denotes the maximum norm, $\delta_i$, $i = 1, 2, 3$, are the least-square stabilization parameters related to the Darcy's law, mass balance and rotational of Darcy's law (Correa and Loula, 2009), respectively, and the stabilization parameters $\beta_S$ and $\beta_D$ are defined as

$$
\beta_S = \frac{\mu \, \beta_0^S}{h} \quad \text{and} \quad \beta_D = A \frac{\beta_0^D}{h} \quad \text{with} \quad \beta_0^S, \beta_0^D > 0. \tag{23}
$$

To solve this problem, the hybrid formulation (18) is splited in a set of local problems defined at the element level and a global problem associated with the multiplier. The degrees of freedom of the interest variables in the local problem are condensed, through the static condensation technique, and a global system is assembled in terms of the multiplier. Then, the global problem is solved leading to the approximate solution of the multiplier, which is plugged into the local problems to recover the discontinuous approximation of the velocity and pressure fields.

### 4.1 Concentration Aproximation

Since calculated the velocity field through the hybrid method (18), we can obtain the concentration field using the SUPG method Brooks and Hughes (1982) to aproximate the transport equation (10)-(12). For this, let the time step $\Delta t > 0$, such that $N = T/\Delta t$ and $t_n = n\Delta t$ with $n = 1, 2, ..., N$ and $I_h = \{0 = t_0 < t_1 < ... < t_N = T\}$ a partition of the interval $I = [0, T]$. Thus, the term involving the time derivative of the concentration is approximated by backward Euler finite difference operator. Therefore, a semi-discrete approximation for the transport equation for each $n = 1, 2, ...N$, given $c^0(\mathbf{x}) = c_0(\mathbf{x})$, can be written as

$$
\phi \frac{c^{n+1} - c^n}{\Delta t} + \mathbf{u} \cdot \nabla c^{n+1} - \operatorname{div}(\mathbf{D}(\mathbf{u}) \nabla c^{n+1}) + \hat{f} c^{n+1} = g^{n+1} \quad \text{in} \quad \Omega. \tag{24}
$$

Combine the semi-discrete approximation (24) with a stabilized finite element method in space (SUPG), and introduce the following fully discrete approximation for the concentration equation: for time levels $n = 1, 2, ...N$, find $c_h^{n+1} \in \mathcal{C}_h^k$, where $\mathcal{C}_h^k$ is a $C^0$ Lagrangean finite element space of degree at most $k$, such that

$$A_{SUPG}(c_h^{n+1}, \varphi_h) = F_{SUPG}(\varphi_h), \quad \forall \varphi_h \in \mathcal{C}_h^k \tag{25}$$

with

$$\begin{aligned} A_{SUPG}(c_h^{n+1}, \varphi_h) &= \phi \int_\Omega c_h^{n+1} \varphi_h \, d\mathbf{x} + \Delta t \int_\Omega \mathbf{u}_h \cdot \nabla c_h^{n+1} \varphi_h \, d\mathbf{x} \\ &+ \Delta t \int_\Omega \mathbf{D} \nabla c_h^{n+1} \cdot \nabla \varphi_h \, d\mathbf{x} + \Delta t \int_\Omega \hat{f} c_h^{n+1} \varphi_h \, d\mathbf{x} \\ &+ \sum_{K \in \mathcal{T}_h} \int_K \left( \phi c_h^{n+1} + \Delta t \mathbf{u}_h \cdot \nabla c_h^{n+1} + \Delta t \hat{f} c_h^{n+1} \right) (\delta_K \mathbf{u}_h \cdot \nabla \varphi_h) \, ds \\ &+ \sum_{K \in \mathcal{T}_h} \int_K \left( -\Delta t \, \mathrm{div}(\mathbf{D} \nabla c_h^{n+1}) \right) (\delta_K \mathbf{u}_h \cdot \nabla \varphi_h) \, ds \end{aligned} \tag{26}$$

$$\begin{aligned} F_{SUPG}(\varphi_h) &= \phi \int_\Omega c_h^n \varphi_h \, d\mathbf{x} + \Delta t \int_\Omega g^{n+1} \varphi_h \, d\mathbf{x} \\ &+ \sum_{K \in \mathcal{T}_h} \int_K \left( \phi c_h^n + \Delta t \, g^{n+1} \right) (\delta_K \mathbf{u}_h \cdot \nabla \varphi_h) \, ds. \end{aligned} \tag{27}$$

In the system (25) the velocity field $\mathbf{u}_h$ is given by the hybrid formulation (18). The stabilization parameter $\delta_K$ depends the Péclet number and is defined in Brooks and Hughes (1982).

## 5. NUMERICAL RESULTS

In this section we present numerical experiments to evaluate the rates of convergence of the hybrid mixed formulation (18). Moreover, we use the approximate velocity field obtained by the hybrid method, which is responsible for the flow displacement, to find the concentrantion field calculated by a predominantly convective equation (10) that is numerically solved via SUPG method Brooks and Hughes (1982) applied to continuous injection processes in a quarter of a repeated five-spot pattern for different heterogenous scenarios.

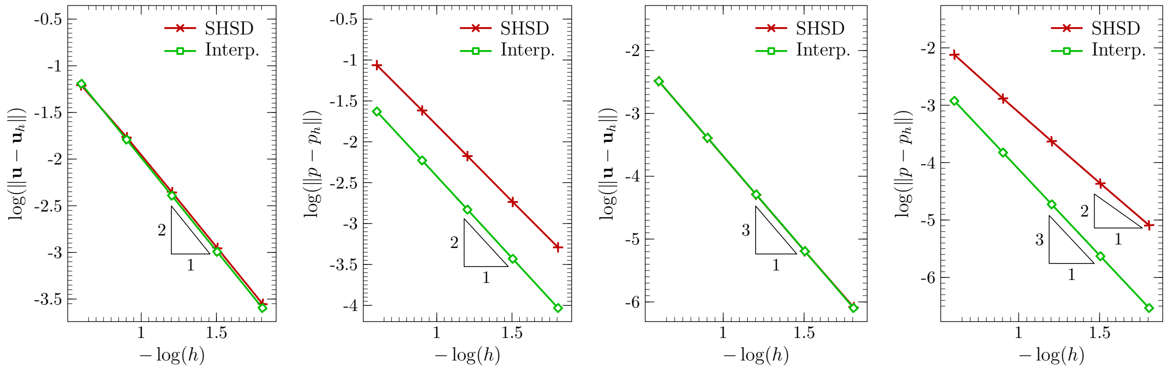
### 5.1 Convergence Study

In this test we solve a simple problem with $\mathbf{K} = \mathbf{I}$ and $\mu = 1.0$ in a square domain $\Omega = \Omega_D \cup \Omega_S = (0, 0; 1, 0) \times (0, 0; 1, 0)$, where $\Omega_D = (0, 0; 1, 0) \times (0, 0; 0, 5)$ and $\Omega_S = (0, 0; 1, 0) \times (0, 5; 1, 0)$, with respectively Stokes and Darcy sources (Correa and Loula, 2009)

$$\mathbf{f} = \begin{bmatrix} (1/2 + 1/(8\pi^2)) \sin(\pi x) \exp(y/2) \\ (\pi - 3/(4\pi)) \cos(\pi x) \exp(y/2) \end{bmatrix}, \qquad f = \left( \frac{1}{2\pi} - 2\pi \right) \cos(\pi x) \exp(y/2),$$

In the convergence study we adopt h-refinement strategy taking a sequence $n \times n$, with $4, 8, 16, 32, 64$, of uniform meshes of quadrilateral elements $\mathbb{Q}_k \mathbb{Q}_l - p_m$, where $k$, $l$ and $m$ denote, respectively, the degree of polynomial space for velocity, pressure and multiplier, considering equal order approximations for all fields $k = l = m = 1$ and 2 with the respectives stabilization parameters for the Stokes and Darcy multipliers $\beta_0^S = 12.0$ and 24.0 and $\beta_0^D = 1.0$ and $15.0$. For the least square stabilization parameters defined in the interior of the elements we adopt in all simulations $\delta_1 = -0.5, \delta_2 = 0.5, \delta_3 = 0.5$.

In Figure 2 we can see the $h$-convergence study for the velocity, pressure and Lagrange multiplier in the $L^2(\Omega)$ norm compared to the interpolant for $\mathbb{Q}_1\mathbb{Q}_1 - p_1$ and $\mathbb{Q}_2\mathbb{Q}_2 - p_2$ elements. The results demonstrate optimal convergence rates for all fields studied, except for the pressure field approximated by biquadratic elements (Fig. 2(d)) in this case the potential loses the convergence rate.

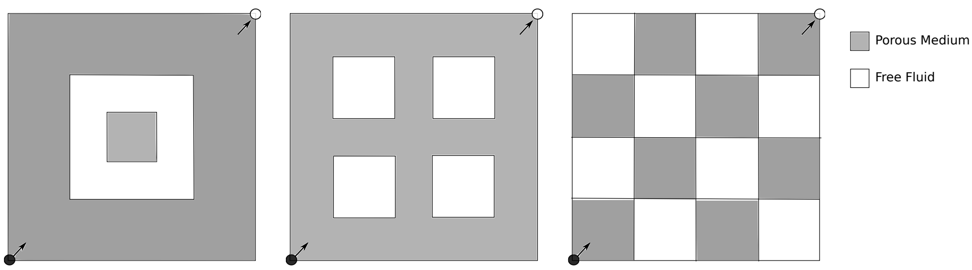



(a) $\mathbf{u}_h$ ($\mathbb{Q}_1\mathbb{Q}_1 - p_1$) (b) $p_h$ ($\mathbb{Q}_1\mathbb{Q}_1 - p_1$) (c) $\mathbf{u}_h$ ($\mathbb{Q}_2\mathbb{Q}_2 - p_2$) (d) $p_h$ ($\mathbb{Q}_2\mathbb{Q}_2 - p_2$)

Figure 2- $h$-convergence study of the $\mathbf{u}_h$ and $p_h$ approximations comparing the hybrid method with the interpolant (Interp.) in $L^2(\Omega)$ norm for $\mathbb{Q}_1\mathbb{Q}_1 - p_1$ and $\mathbb{Q}_2\mathbb{Q}_2 - p_2$ elements.

## 5.2 Continuous Injection Simulation

In here we simulate a quarter of a repeated five-spot pattern in two dimension consisting of a square domain (unit thickness) with side $L = 1000.0\,ft$. The injector well is located at the lower-left corner ($x = y = 0$) and the producer well at the upper-right corner ($x = y = L$). For this, we use the hybrid formulation to aproximate the hydrodynamic problem, then we supply the velocity field to the transport equation that is numerically solved by the SUPG method combined with an implicit finite difference scheme in three different scenarios described in Figure 3.

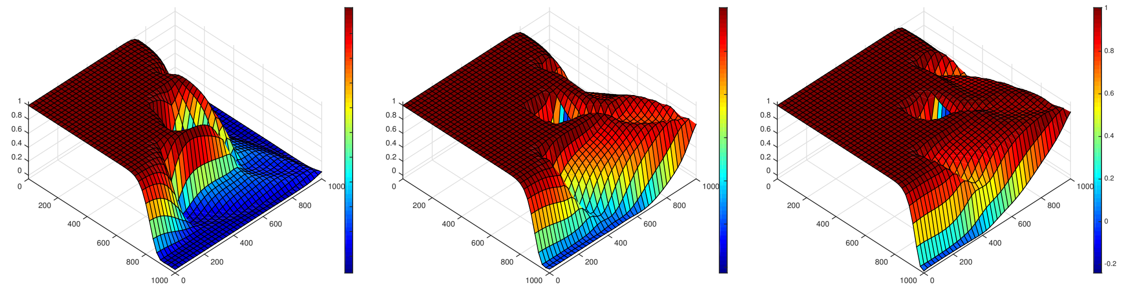



(a) Scenario 1 (b) Scenario 2 (c) Scenario 3

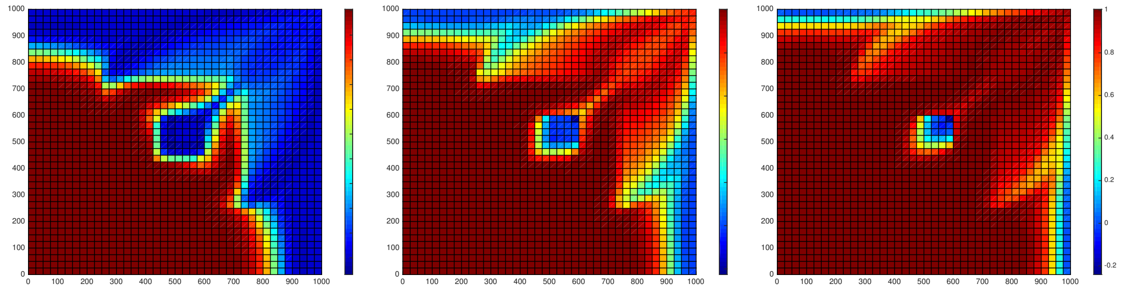
Figure 3- Three coupled porous medium (shaded) free fluid (white) domains used to simulate the five-spot problem.

In three cases is considered a porous medium with homogeneous permeability $\kappa = 10.0mD$, where $\mathbf{K} = (\kappa/\mu)\mathbf{I}$, viscosity of the resident fluid is $\mu = 1.0\,cP$, porosity $\phi = 0.1$, molecular diffusion $\alpha_m = 0.0$, longitudinal dispersion $\alpha_l = 10.0\,ft^2/day$, transverse dispersion $\alpha_t = 1.0\,ft^2/day$ and the flow rate are 800 square feet per day. For the Stokes region the diffusion tensor is chosen to be $\mathbf{D} = \alpha_m\mathbf{I}$ with $\alpha_m = 1.0\,ft^2/day$ . We use the same values of the stabilization parameters $\delta_1 = -0.5, \delta_2 = 0.5, \delta_3 = 0.5$ and $\alpha = 1.0$. Moreover, a time step of 5 days and uniform meshes of $40 \times 40$ bilinear quadrilateral elements are adopted employing equal order approximations for all fields (velocity, pressure, multiplier and concentration).

The Figures 4-6 show the concentration maps and concentration contours for the proposed scenarios. In these graphs we can clearly observe the effect of the barrier on the continuous injection transport generated by the low permeability of the porous medium. The continuous injection concentration in the scenario 3 (Fig. 6) takes longer time to reach the producer well due to the higher heterogeneity of the medium, because presents more discontinuities generated by the free fluid/porous medium interfaces, which reduces the flow velocity.



(a) 1000 days (b) 1500 days (c) 2000 days



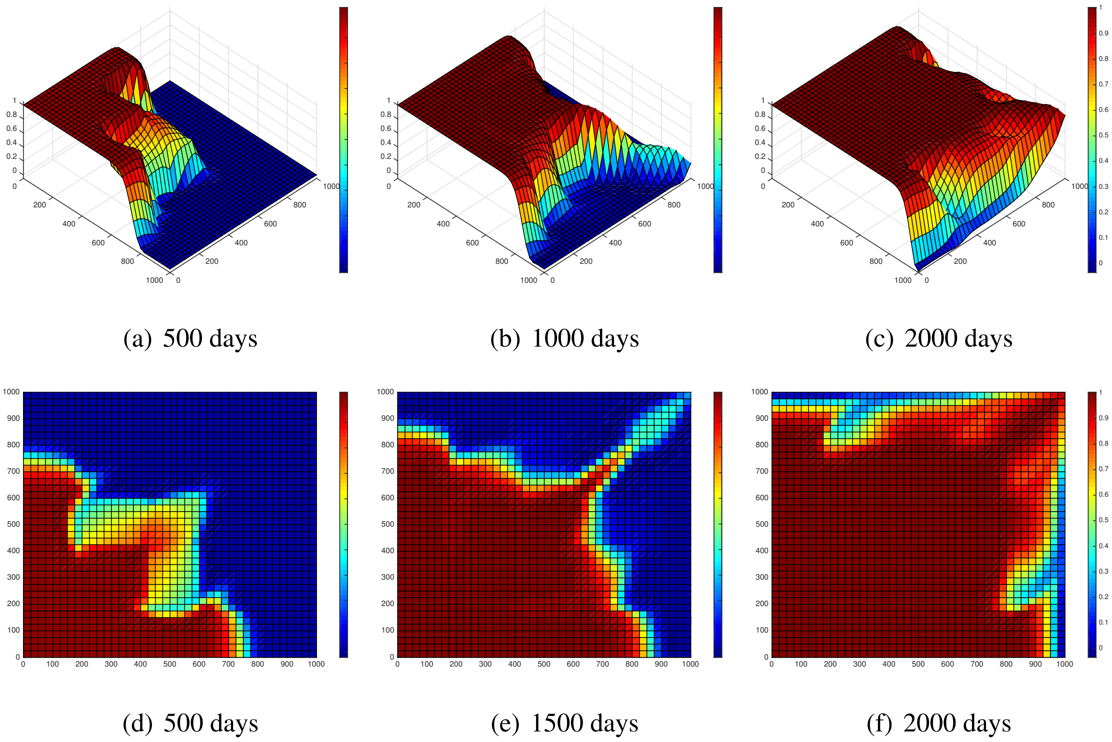
(d) 1000 days (e) 1500 days (f) 2000 days

Figure 4- Scenario 1: propagation of the concentration in five-spot problem at 1000, 1500 and 2000 days.

## 6. CONCLUSIONS

In this work we recalled the hybrid formulation for the Stokes-Darcy problem introduced by Igreja and Loula (2018) to solve the hydrodynamic flow of the transport concentrantion aproximated by SUPG method combined with a backward finite difference scheme in time to simulate the problems in free fluid/porous medium domain. The hybrid method imposes naturally the interface conditions due to use the Lagrange multipliers not only on the interface, but in all domain.

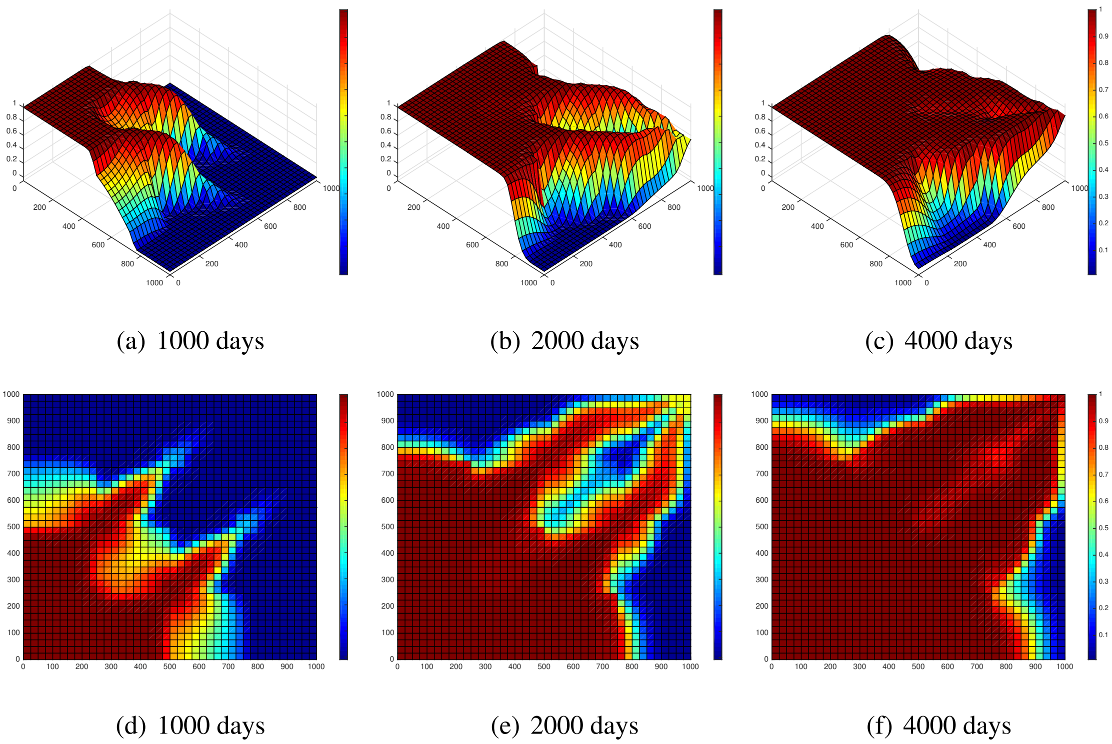
The convergence results for the hybrid method illustrate the flexibility and robustness of the hybrid finite element formulation and show optimal rates of convergence to the velocity field. With respect the concentration approximation, the combination hybrid/SUPG gave stable and accurate results in heterogeneous media formed by free fluid and porous medium capturing precisely the phenomena arising of this interaction.



(a) 500 days (b) 1000 days (c) 2000 days

(d) 500 days (e) 1500 days (f) 2000 days

Figure 5- Scenario 2: propagation of the concentration in five-spot problem at 500, 1000 and 2000 days.



(a) 1000 days (b) 2000 days (c) 4000 days

(d) 1000 days (e) 2000 days (f) 4000 days

Figure 6- Scenario 3: propagation of the concentration in five-spot problem at 1000, 2000 and 4000 days.

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